IOSR Journal of Engineering (IOSRJEN) **www.iosrjen.org**
ISSN (e): 2250-3021, ISSN (p): 2278-8719
Vol. 06, Issue 06(June. 2016), ||V2|| PP 18-26

Significance of Dirichlet Series Solution for a Boundary Value Problem

Achala L. Nargund* and Pramod S.*
P.G. Department of Mathematics and Research Centre in Applied Mathematics, M.E.S College of Arts, Commerce and Science, Bengaluru-560003

Abstract: Dirichlet series is an exponential power series, introduced by P. G. L. Dirichlet a German mathematician. Using this power series, we study the boundary-layer equation of flow over a nonlinearly stretching sheet in the presence of magnetic field. The solution obtained by this method is in good agreement with existing solutions.

Keywords: *Boundary Layer Equations, Adomian decomposition method, Homotopy Analysis Method, Dirichlet Series, Newton-Raphson Method.*

I. INTRODUCTION

The boundary layer flow caused by a continuously stretching sheet is often encountered in many engineering and industrial processes. These flows are existing in the polymer industry, in the manufacture of sheeting material through an extrusion process and cooling of an infinite metallic plate. Most boundary layer models can be reduced to systems of nonlinear ordinary differential equations which are usually solved either by numerical methods or by analytic methods. Analytic methods have significant advantages over pure numerical methods in providing more convergent solutions.

In the modern developments of the theory of power series a great part has been played by a variety of methods of summation of oscillating series, which we associate with the names of Frobenius, Holder, Cesaro, Borel, Lindelof, MittagLeffler and Le Roy. Of these definitions the simplest and the most natural is that which defines the sum of an oscillating series as the limit of the arithmetic mean of its first n partial sums. This definition was generalized in two different ways by Holder and by Cesaro, who thus arrived at two systems of definitions the complete equivalence of which has been established only recently by Knopp, Schnee, Ford and Schur.

The range of application of Cesaro's methods is limited in a way which forbids their application to the problem of the analytical continuation of the function represented by a Taylor's series. A power series, outside its circle of convergence, diverges too crudely for the application of such methods: more powerful, though less delicate, methods such as Borel's are required. But Cesaro's methods have proved of highest value in the study of power series on the circle of convergence and closely connected problems of the theory of Fourier series. And it is natural to suppose that in the theory of Dirichlet Series, where we are dealing with series whose convergence or divergence is of a much more delicate character than is, in general, that of power series, they will find a wider field of application [5], [6].

Dirichlet series solution is an exponential series solution, which are most useful, especially in obtaining derived quantities, than pure numerical schemes. The accuracy and the uniqueness of the solution so obtained can be confirmed using other equally powerful semi numerical schemes. This method explained by P. L. Sachdev [9]. Sachdev. et. al (2000) proposed the Dirichlet series solution approach to uniform flow past a semi-infinite flat plate. Here they have found the solution of free convention flows in saturated porous media and also solution of Falkner-Skan equation in the axisymmetric flow due to stretching flat surface [10]. Sachdev. et. al (2005) proposed the Dirichlet series solutions to the boundary value problems for third order nonlinear ordinary differential equations over an infinite interval. In this paper they found the solution of Falkner-Skan equation occurs in the magneto hydrodynamics [11].

Awang. et. al [1] have studied the boundary layer equation of flow over a nonlinearly stretching sheet in the presence of chemical reaction and a magnetic field using Adomian Decomposition Method (ADM). S. B. Sathyanarayana and L. N. Achala [2] investigated the solution by Homotopy Analysis Method (HAM) for the boundary layer flow of a viscous flow over a nonlinearly stretching sheet in the absence of a chemical reaction and in presence of magnetic field. We are interested in applying the Dirichlet Series Solution method to obtain an approximate analytic solution of the boundary layer viscous flow over a nonlinearly stretching sheet in the absence of a chemical reaction and in presence of magnetic field. Comparison of the present solution with the solutions obtained by Awang. et. al [1] in the absence of a chemical reaction and S. B. Sathyanarayana and L. N. Achala [2] by HAM were also made.

II. GOVERNING EQUATIONS

Here we consider the steady two-dimensional incompressible flow of an electrically conducting viscous fluid past a nonlinearly semi-infinite stretching sheet under the influence of a constant transverse applied magnetic field. The magnetic Reynolds number is assumed small and negligible in comparison to the applied magnetic field. The governing boundary layer equations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=v_m\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}u, \qquad (2)$$

where x and y are distances along and perpendicular to the sheet respectively, u and v are component of the velocity along x and y directions respectively, v_m kinematic viscosity, ρ fluid density, σ electrical conductivity, B_0 strength of the magnetic field. The corresponding boundary conditions for nonlinear stretching sheet are as follow,

$$u(x,0)=ax+cx^2, v(x,0)=0, \qquad (3)$$

$$u\to 0 \; as \; y\to\infty, \qquad (4)$$

where d and c are constants.

We introduce the similarity transformations [1],

$$\eta=\sqrt{\frac{a}{v_m}}y, \quad u=axf'(\eta)+cx^2g'(\eta), \qquad (5)$$

$$v=-\sqrt{av_m}\,f(\eta)-\frac{2cx}{\sqrt{a/v_m}}g(\eta), \quad N=\frac{\rho B_0^2}{a\rho}, \qquad (6)$$

substituting equations (5) and (6) in (1) and (2) we obtain the following ordinary differential equations,

$$f'''+ff''-(f')^2-Nf'=0, \qquad (7)$$

$$g'''+fg''-3f'g'+2f''g-Ng'=0, \qquad (8)$$

subject to boundary conditions (3) and (4),

$$f(0)=0,\; f'(0)=1,\; f'(\infty)=0, \qquad (9)$$

$$g(0)=0,\; g'(0)=1,\; g'(\infty)=0, \qquad (10)$$

where f and g are functions related to the velocity filed, N is a magnetic parameter and the primes denote differentiation with respect to η .

III. DIRICHLET SERIES APPROACH TO BOUNDARY VALUE PROBLEM OVER AN INFINITE INTERVAL

We assume the Dirichlet series solutions to equations (7) and (8) satisfying $f'(\infty) = 0$ and $g'(\infty) = 0$ [3], [4] in the form,

$$f(\eta) = \gamma_1 + \gamma \sum_{i=1}^{\infty} b_i d^i e^{-i\gamma\eta}, \tag{11}$$

$$g(\eta) = \gamma_2 + \gamma \sum_{i=1}^{\infty} l_i p^i e^{-i\gamma\eta}, \tag{12}$$

where $\gamma > 0$, d and p are parameters.

Substituting (11) and (12) in (7) and (8), we get

$$\sum_{i=1}^{\infty} [i^3\gamma^2 - \gamma_1\gamma\, i^2 - N i] b_i d^i e^{-i\gamma\eta} - \gamma^2 \sum_{i=2}^{\infty}\sum_{k=1}^{i-1} [2k^2 - ik] b_k b_{i-k} d^i e^{-i\gamma\eta} = 0, \tag{13}$$

and

$$\sum_{i=1}^{\infty} [i^3\gamma^2 - i^2\gamma^2 + i^2 N - N i] l_i p^i e^{-i\gamma\eta} - 2\gamma\gamma_2 \sum_{i=1}^{\infty} i^2 b_i d^i e^{-i\gamma\eta}$$
$$-\gamma^2 \sum_{i=2}^{\infty}\sum_{k=1}^{i-1} [6k^2 - 3ik] b_k l_{i-k} d^i p^{i-k} e^{-i\gamma\eta} = 0, \tag{14}$$

For $i = 1$, equations (13) and (14) reduces to,

$$[\gamma^2 - \gamma_1\gamma - N] b_1 d e^{-\gamma\eta} = 0, \tag{15}$$

$$2\gamma\gamma_2 b_1 d = 0. \tag{16}$$

To avoid trivial solution we set, $|b_1| = 1, |l_1| = 1$.
Hence from equations (15) and (16) we can write,

$$\gamma_1 = \frac{\gamma^2 - N}{\gamma}, \tag{17}$$

$$\gamma_2 = 0, \tag{18}$$

where $\gamma \neq 0, b_1 \neq 0$ and $d \neq 0$.
Substituting equations (17) and (18) in equations (13) and (14) we obtain recursive relations for the coefficients b_i and l_i $(i = 2, 3, 4, \ldots.)$,

$$b_i = \frac{\gamma^2}{i(i-1)(i\gamma^2 + N)} \sum_{k=1}^{i-1} [2k^2 - ik] b_k b_{i-k}, \tag{19}$$

and

$$l_i = \frac{3\gamma^2}{i(i-1)(i\gamma^2 + N)} \sum_{k=1}^{i-1} [2k^2 - ik] b_k l_{i-k} \left(\frac{d}{p}\right)^k. \tag{20}$$

If the series (11) and (12) converge absolutely when $\gamma > 0$ for some η_0, then these series converge absolutely and uniformly in the half plane $Re(\eta) \geq Re(\eta_0)$ such that $f'(\infty) = 0$ and $g'(\infty) = 0$ [3], [4]. A general discussion of the convergence, etc., of the Dirichlet series (11) and (12) can be found in [5], [6].
The unknown parameters γ, d, p are determined from the remaining boundary conditions (9) and (10) at

$\eta = 0$ as follows,

$$f(0) = \frac{\gamma^2 - N}{\gamma} + \gamma \sum_{i=1}^{\infty} b_i d^i, \tag{21}$$

$$f'(0) = \gamma^2 \sum_{i=1}^{\infty} (-i) b_i d^i - 1, \tag{22}$$

$$g(0) = \gamma \sum_{i=1}^{\infty} l_i p^i, \tag{23}$$

$$g'(0) = \gamma^2 \sum_{i=1}^{\infty} (-i) l_i p^i - 1. \tag{24}$$

The numerical values of γ, d and p are calculated by applying Newton - Raphson method to equations (21), (22) and (24) (Table 1).

Substituting equations (17) and (18) in equations (11) and (12), then the Dirichlet Series solutions for (7) and (8) will be,

$$f(\eta) = \frac{\gamma^2 - N}{\gamma} + \gamma \sum_{i=1}^{\infty} b_i d^i e^{-i\gamma\eta}, \tag{25}$$

$$g(\eta) = \gamma \sum_{i=1}^{\infty} l_i p^i e^{-i\gamma\eta}, \tag{26}$$

where b_i and l_i can be calculated by using equations (19) and (20).

The shear stress at the surface is given by

$$f''(0) = \gamma^3 \sum_{i=1}^{\infty} b_i d^i. \tag{27}$$

The exact solution of (7) subject to boundary conditions (9) is [16],

$$f(\eta) = \frac{1}{\sqrt{1+N}} [1 - exp(-\sqrt{1+N}\,\eta)], \tag{28}$$

$$f''(0) = -\sqrt{1+N}. \tag{29}$$

Obtained Dirichlet series solution (25) and exact solution (28) are compared graphically for the different values of N in Figure 1. The exact shear stress given in (29) is compared numerically with our Dirichlet series shear stress (29) in Table 1.

IV. NUMERICAL RESULTS AND CONCLUSIONS

In the present paper, we have given approximate analytic solution of the boundary layer viscous flow over a nonlinearly stretching sheet in the absence of a chemical reaction and in presence of magnetic field in the form of Dirichlet series (25) and (26). The calculated values of $f''(0)$ representing shear stress at the surface for the different sets of values of N, d and γ are given in Table 1. Comparison of the values obtained by the Dirichlet series method with other methods given in Table 2. The velocity $f'(\eta)$ of exact solution (28) and Dirichlet series solution (25) are compared graphically for the different values of N in Figure 2.

Figures 3, 5, 7, 9 represents the component of velocity $u(x, y)$ (5) and Figures 4, 6, 8, 10 represents the component of velocity $v(x, y)$ (6) for the different values N. Both $u(x, y)$ and $v(x, y)$ are obtained by substituting Dirichlet series (25) and (26) in equations(5) and (6). We can observe that the component of velocity $v(x, y)$ moving upwards as N increases .

The solution obtained by using the Dirichlet series method matches with the solution obtained by using Adomian Decomposition Method [1], Homotopy Analysis Method [2] and the exact solution [16]. We conclude that the Dirichlet series method is a strong analytic method to solve nonlinear differential equations and it produces good convergent solution.

Table 1: Comparison of the values of $f''(0)$ obtained by the Dirichlet series method and exact solution for the different values of N, γ and d.

N	γ	d	$f''(0)$ Dirichlet	$f''(0)$ Exact
0	1	-1	-1	-1
1	1.41421365	-0.50000022	-1.41421445	-1.41421356
1.5	1.58115203	-0.40003107	-1.58130125	-1.58113883
3	2.00001593	-0.25003261	-2.00030867	-2.00000000
9	3.16227767	-0.10000001	-3.16227793	-3.16227766
25	5.09902318	-0.03846502	-5.09949184	-5.09901951

Table 2: Comparison of the values of $f''(0)$ obtained by using the Dirichlet Series Method and other methods.

N	Dirichlet Series	Exact	ADM	HAM
0	-1	-1	-1	-1
1.5	-1.581301	-1.58113883	-1.58113908	-1.544886
9	-3.1622779	-3.16227766	-3.16227766	-3.1595

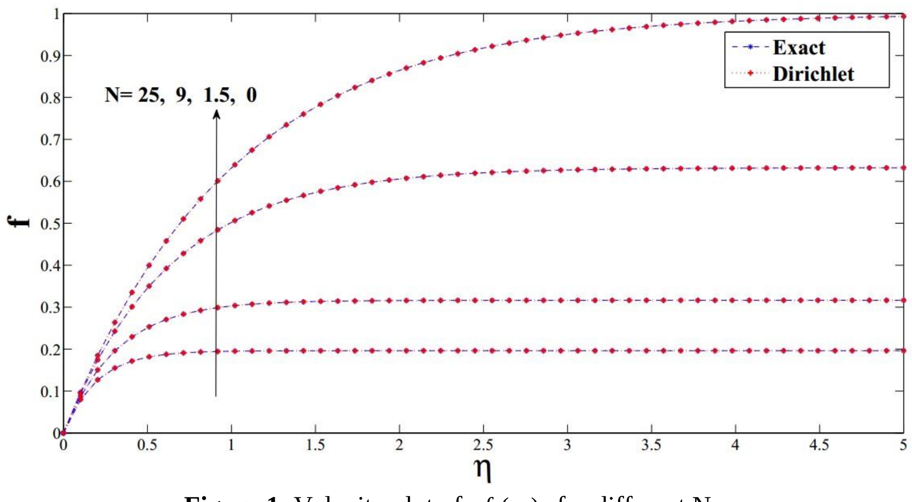

Figure 1: Velocity plot of $f(\eta)$ for different N

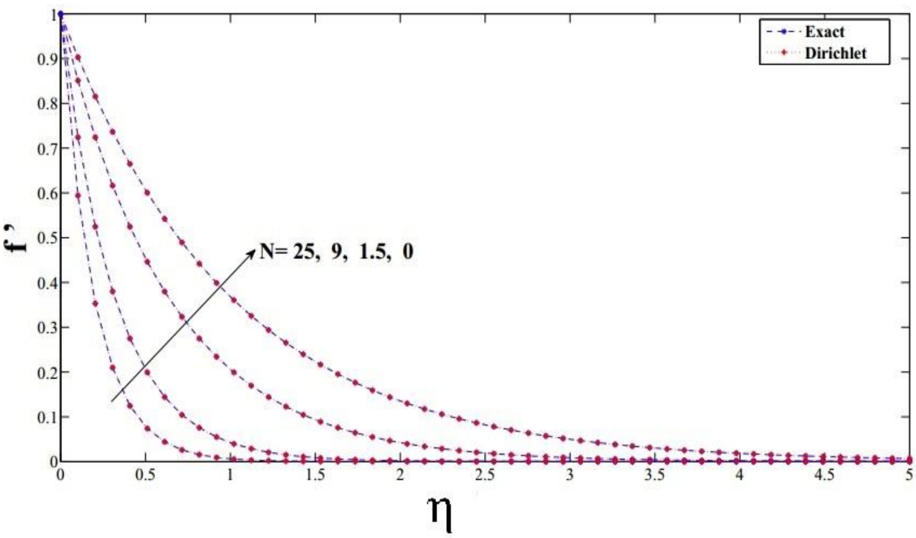

Figure 2: Velocity plot of $f\,'(\eta)$ for different N

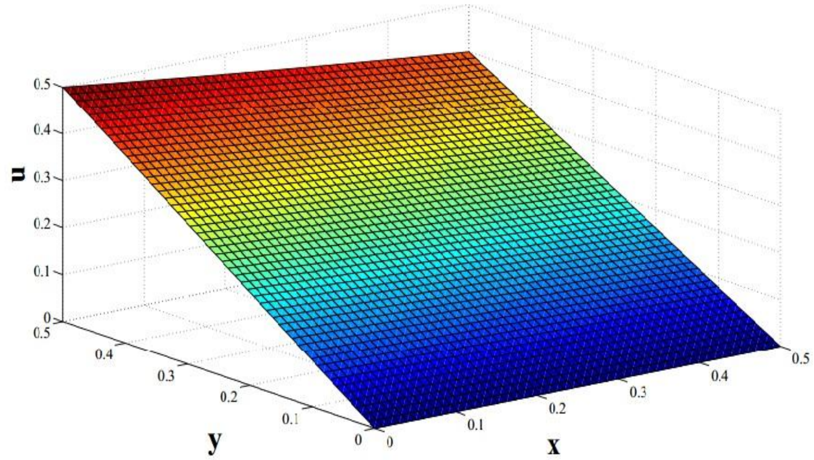

Figure 3: $u(x, y)$ for $N = 0$

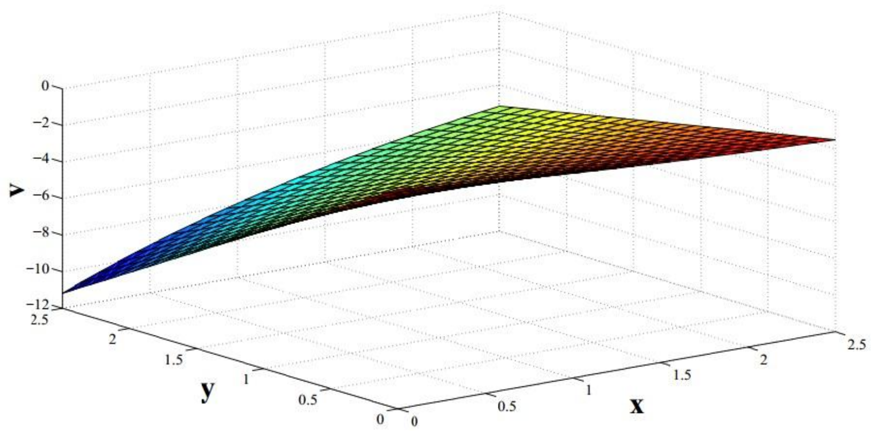

Figure 4: $v(x, y)$ for $N = 0$

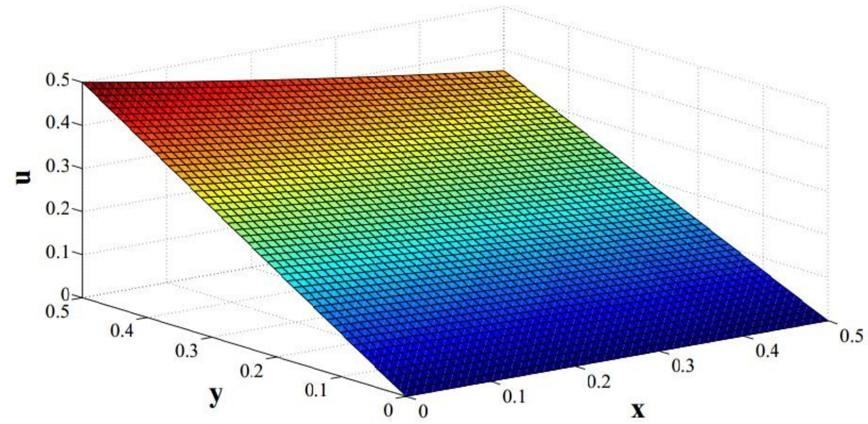

Figure 5: $u(x, y)$ for $N = 1.5$

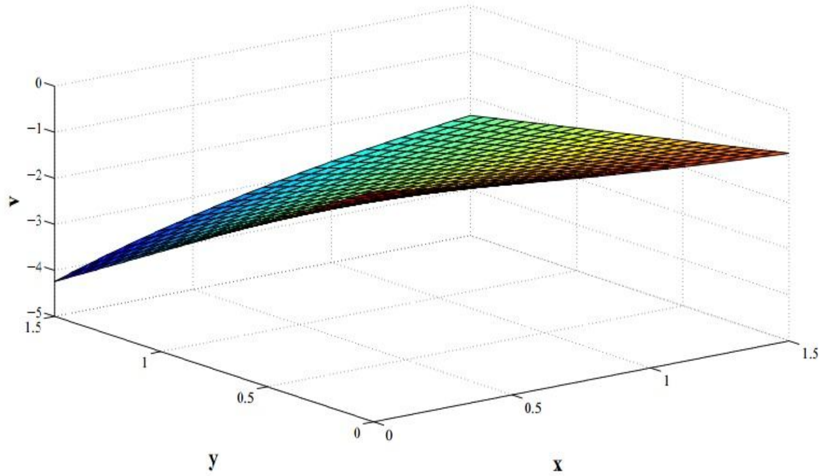

Figure 6: $v(x, y)$ for $N = 1.5$

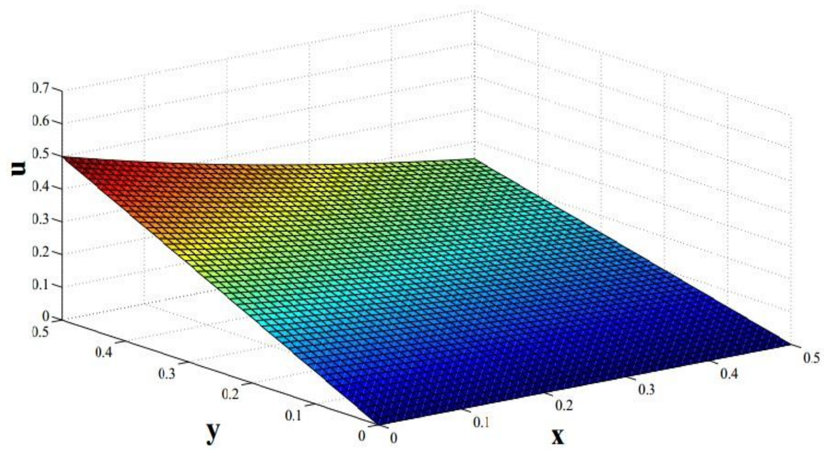

Figure 7: $u(x, y)$ for $N = 9$

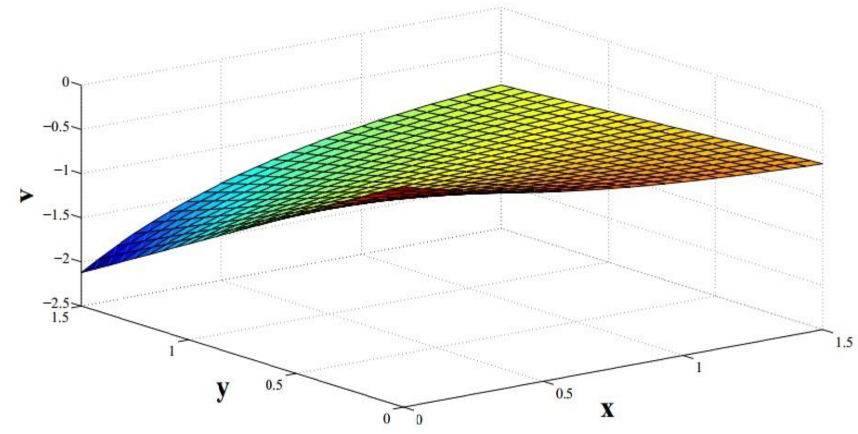

Figure 8: $v(x, y)$ for $N = 9$

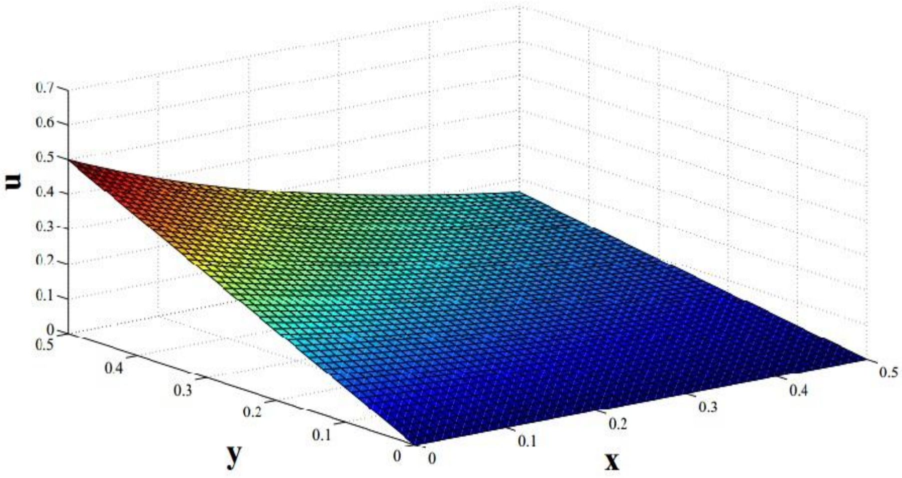

Figure 9: $u(x, y)$ for $N = 25$

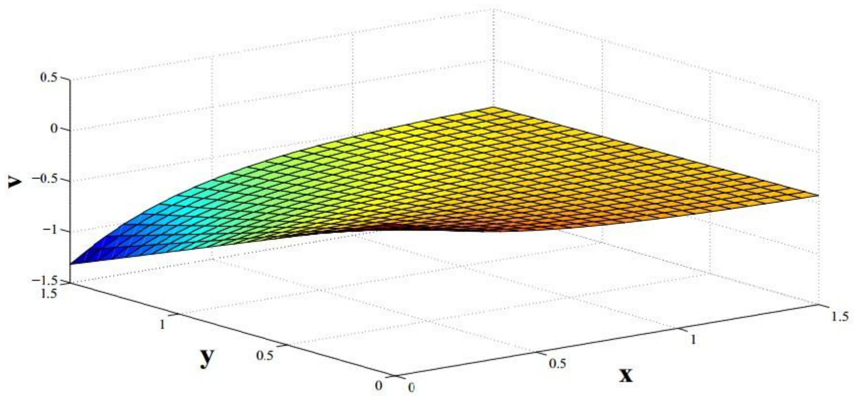

Figure 10: $v(x, y)$ for $N = 25$

V. ACKNOWLEDGEMENT

The authors are thankful to Vision Group of Science and Technology (VGST) for providing financial assistance through GRD 105, CISE.

REFERENCES

[1] S. Awang Kechil and I. Hashim, Series solution of flow over nonlinearly stretching Sheet with chemical reaction and magnetic field, Physics Letters A 372, 2258-2263, 2008.
[2] S. B. Sathyanarana and L. N. Achala 2011, Homotopy analysis method for flow over nonlinearly stretching sheet with magnetic Field, Journal of Applied Mathematics and Fluid mechanics, ISSN 09743170 Volume 3, Number 1, pp., 15-22, 2011.
[3] T.K.Kravchenko, A.I.Yablonskii, Solution of an infinite boundary value problem for third order equation, Differential'nye Uraneniya, vol. 1,pp. 327, 1965.
[4] T.K. Kravchenko, A.I.Yablonskii, A boundary value problem on a semi-infinite interval, Differential'nye Uraneniya, vol. 8(12), pp. 2180-2186, 1972.
[5] S.Riesz , Introduction to Dirichlet series, Camb. Univ. Press, 1957.
[6] G.H. Hardy, M. Riesz, The general theory of Dirichlet's series, Cambridge Univ. Press, 1915.
[7] J.H. Merkin, A note on the solution of a differential equation arising in boundary layer theory, J. Engg. Mathematics 18, 31, 1984.
[8] P.L. Sachdev, Nonlinear ordinary differential equations and their applications, Marcel Dekker, INC, 1991.
[9] P.L. Sachdev, N. M. Bujurke and N. P. Pai, Dirichlet series solution of equations arising in boundary layer theory, Mathematical and computer modelling 32 , 971-980, 2000.
[10] V. M. Falkner and S. W. Skan, Solutions of the boundary layer equations, Phil. Soc. Mag. 12: 865, 1931.
[11] P. L. Sachdev, N. M. Bujurke and V. B. Awati, Boundary value problems for third order nonlinear ordinary differential equations, Studies in Applied Mathematics, Vol. 115, No. 3, pp. 303-318, 2005.
[12] N. M. Bujurke, R. B. Kudenatti and V. B. Awati, Dirichlet series solution of mhd flow over a nonlinear stretching sheet, International journal of applied mathematics and engineering sciences, Vol. 5, No 1, January-June 2011.
[13] N. M. Bujurke, R. B. Kudenatti and V. B. Awati, An exponential series method for the solution of free convection boundary layer flow in a saturated porous medium, American journal of computational mathematics, 1, 104-110, 2011.
[14] N. M. Bujurke and V. B. Awati, Approximate analytical solutions of MHD flow of a viscous fluid on a nonlinear porous shrinking sheet, Bulletin of the international mathematical virtual institute, vol. 4, 145-155, 2014.
[15] R. B.Kudenatti, V. B.Awati and N. M.Bujurke, Exact analytical solutions of class of boundary layer equations for a stretching surface, Appl Math Comp, vol. 218, pp. 2952- 2959, 2011.
[16] F. T. Akyildiz, H. Bellout, K. Vajravelu, J. Math. Anal. Appl. 320, 322, 2006.